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The first eigenvalue of the Dirac operator on Kähler manifolds

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Abstract. *If M^{2m} is a closed Kähler spin manifold of positive scalar curvature R, then each eigenvalue λ of type r ($r \in \{1, \dots, [(m+1)/2]\}$) of the Dirac operator D satisfies the inequality $\lambda^2 \geq rR_0/4r-2$, where R_0 is the minimum of R on M^{2m}. Hence, if the complex dimension m is odd (even) we have the estimation $\lambda \geq \sqrt{(m+1)R_0/4m}$ ($\lambda \geq \sqrt{mR_0/4(m-1)}$) for the first eigenvalue of D. In the paper is also considered the limiting case of the given inequalities. In the limiting case with $m = 2r-1$ the manifold M^{2m} must be Einstein. The manifolds $S^2, S^2 \times S^2, S^2 \times T^2, P^3(\mathbb{C}), F(\mathbb{C}^3), P^3(\mathbb{C}) \times T^2$ and $F(\mathbb{C}^3) \times T^2$, where $F(\mathbb{C}^3)$ denotes the flag manifold and T^2 the 2-dimensional flat torus, are examples for which the first eigenvalue of the Dirac operator realizes the limiting case of the corresponding inequality. In general, if M^{2m} is an example of odd complex dimension m, then $M^{2m} \times T^2$ is an example of even complex dimension $m+1$. The limiting case is characterized by the fact that here appear eigenspinors of D^2 which are Kählerian twistor-spinors.*

INTRODUCTION

From [6] it is known that any eigenvalue λ of the Dirac operator D on a closed Kähler spin manifold M^n with positive scalar curvature R satisfies the inequality

$$(*) \qquad \lambda^2 \geq \frac{1}{4}\frac{m+1}{m}R_0$$

where $m = n/2$ is the complex dimension and R_0 the minimum of R on M^n. Moreover, if $\sqrt{(m+1)R_0/4m}$ itself is an eigenvalue of D, then M^n is an Einstein

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space of odd complex dimension m. Estimation (*) is sharp in the sense that there are Kähler manifolds with this property. For the complex dimensions $m = 1$ and $m = 3$ we have a complete list of such spaces, namely for $m = 1$ the sphere $s^2 = P^1(\mathbb{C})$ and for $m = 3$ the complex projective space $P^3(\mathbb{C})$ and the flag manifold $F(\mathbb{C}^3)$ with respect to their natural Kähler structures (cf. [4], [8]). According to this, for even complex dimension the inequality (*) can never be an equality for the first eigenvalue. It is a conjecture that in this case it may be possible to obtain a better estimation than (*). In fact, here we prove that for even complex dimension m each eigenvalue λ of the Dirac operator D satisfies the inequality

$$\lambda^2 \geq \frac{1}{4}\frac{m}{m-1}R_0 . \tag{2*}$$

This estimation is sharp, too. For example, in dimension $n = 4$ for the Grassmannian manifold $G_{2,4} = S^2 \times S^2$ we have $R = 8$ and the first eigenvalue of D^2 is equal to 4 (cf. [11]). Another example is $S^2 \times T^2$, where here and in the following T^2 denotes the flat torus. Further, in dimension $n = 8$ we have the examples $P^3(\mathbb{C}) \times T^2$ and $F(\mathbb{C}^3) \times T^2$. We prove in general that if M^n is an example of odd complex dimension $m = n/2$, then $M^n \times T^2$ is an example of even complex dimension $m + 1$. The inequalities (*), (2*) are corollaries of a more general result saying that each eigenvalue λ of type r ($r \in \{1, \ldots, [(m+1)/2]\}$) of the Dirac operator satisfies the inequality

$$\lambda^2 \geq \frac{r}{4r-2}R_0 . \tag{3*}$$

This can be interpreted as follows: let S be the spinor bundle and L a holomorhic line bundle corresponding to the given spin structure on M^n. From [5] it is known that there is a natural isomorphism $S = \oplus_{r=0}^{m} \Lambda^{0,r} \otimes L$ and that via this isomorphism D^2 can be identified with the restriction $\Delta_L^{0,\cdot} = \oplus_{r=0}^{m} \Delta_L^{0,r}$ of the Laplacian Δ_L of the Dolbeault comples of L. Then each eigenvalue μ of the Laplacian $\Delta_L^{0,r} : \Gamma(\Lambda^{0,r} \otimes L) \to \Gamma(\Lambda^{0,r} \otimes L)$ with $r \in \{1, \ldots, [(m+1)/2]\}$ satisfies the inequality $\mu \geq rR_0/(4r-2)$. Moreover, the spectra of $\Delta_L^{0,0}$ and $\Delta_L^{0,1}$ are equal and also for $r = 0, \ldots m$ the spectra of $\Delta_L^{0,r}$ and $\Delta_L^{0,m-r}$. An essential point in the proof of the inequalities (3*) are Weitznböck formulas, in which the Kählerian twistor operators $\mathscr{D}^r$ of type r ($r = 1, \ldots, m$) are involved. In the limiting case of inequality (3*), eigenspinors of D^2 which are elements of ker $\mathscr{D}^r$ appear. The elements of ker $\mathscr{D}^r$ are called «Kählerian twistor-spinors of type r». We remark that twistor-spinors in case of an arbitrary Riemannian spin manifold were investigated by A. Lichnerowicz (cf. [9], [10]) and T. Friedrich (cf. [3]).

1. THE KÄHLERIAN TWISTOR OPERATORS

Let (M^n, g, J, S) be a Kähler spin manifold of complex dimension $m = n/2$ with Riemannian metric g, complex structure J and spinors bundle S. The we have two operators $D, \tilde{D} : \Gamma(S) \to \Gamma(S)$ which, with respect to any local frame of vector fields $(X_1, \dots, X_n)$ are locally defined by

$$D\psi = X^k \nabla_{X_k} \psi \qquad \tilde{D}\psi = J(X^k) \nabla_{X_k} \psi ,$$

where here and in the following Einstein's convention of summation is used and $X^k := g^{ik} X_i$, $(g^{ik}) := (g_{ik})^{-1}$, $g_{ik} = g(X_i, X_k)$. ∇ is the usual covariant derivative on S induced by the covariant derivative on M^n (Levi-Civita connection), which we also denote by ∇ in the following. Then D is the Dirac operator of the corresponding Riemannian spin manifold (M^n, g, S). D and $\tilde{D}$ satisfy the operator equations (cf. [5])

$$\tilde{D} = D^2 \qquad D\tilde{D} + \tilde{D}D = 0 \tag{1}$$

and by A. Lichnerowicz we have the well-known formula

$$D^2 = \nabla^* \nabla + \frac{R}{4} \tag{2}$$

where F is the scalar curvature and $\nabla^*\nabla$ the Bochner Laplacian on S. For $r = 1, \dots, m$ and any vector field $X \in \Gamma(TM^n)$ let $\mathscr{D}^r_X : \Gamma(S) \to \Gamma(S)$ be the operator defined by

$$\mathscr{D}^r_X \psi = \nabla_X \psi + \frac{1}{4r}(X D\psi + J(X)\tilde{D}\psi) .$$

Then the operator $\mathscr{D}^r : \Gamma(S) \to \Gamma(TM^n \otimes S)$ locally given by

$$\mathscr{D}^r \psi = X^k \otimes \mathscr{D}^r_{X_k} \psi \qquad (r = 1, \dots, m)$$

is called the Kählerian twistor operator of type r.

Now we recall that the spinor bundle S of a Kähler spin manifold $M^n (n = 2m)$ splits into the orthogonal direct sum

$$S = S_0 \oplus S_1 \oplus \dots \oplus S_m \tag{3}$$

of holomorphic subbundles S_r with $\operatorname{rank}_{\mathbb{C}} S_r = \binom{m}{r}$. This decomposition is induced by the Kähler form Ω defined by $\Omega(X,Y) = g(X, J(Y))$ if Ω is considered an endomorphism of S in the usual sense. If $p_0, \dots, p_m : S \to S$ are the projection corresponding to decomposition (3), then in the sense of an endomorphism of S, we have

$$\Omega = \sum_{r=0}^{m} i(m - 2r) p_r . \tag{4}$$

With respect to any local frame $(X_1, \dots, X_n)$, Ω has the representation

(5) $$\Omega = \frac{1}{2} J(X^k) X_k = -\frac{1}{2} X^k J(X_k) .$$

Using the relations (cf. [5])

(6) $$X\Omega - \Omega X = 2J(X) \qquad J(X)\Omega - J(X)\Omega = -2X$$

(7) $$D\Omega - \Omega D = 2\tilde{D} \qquad \tilde{D}\Omega - \Omega\tilde{D} = -2D$$

a straightforward calculation proves that, for $r = 1, \dots, m$ and each vector field X the equation

$$\Omega \mathscr{D}^r_X = \mathscr{D}^r_X \Omega$$

is satisfied. This, together with equation (4), yields

$$p_s \mathscr{D}^r_X = \mathscr{D}^r_X p_s \qquad (r = 1, \dots, m;\ s = 0, \dots, m) .$$

We introduce the denotation $\mathscr{D}^{r,s}_X = \mathscr{D}^r_X \circ p_s$. Then the corresponding operator

$$\mathscr{D}^{r,s} := \mathscr{D}^r \circ p_s : \Gamma(S) \to \Gamma(TM^n \otimes S)$$

is called the Kählerian twistor operator of type (r, s). We have

$$\mathscr{D}^r = \sum_{s=0}^{m} \mathscr{D}^{r,s} \quad (r = 1, \dots, m) .$$

Furthermore, we recall that the spinor bundle S of a spin manifold M^n of even dimension $n = 2m$ is furnished with a j-structure, i.e. an antilinear bundle map $j : S \to S$ with the properties $\nabla j = 0$, $j^2 = (-1)^{m(m+1)/2}$, $Xj = jX$ and $\langle j\varphi, j\psi\rangle = \langle \psi, \varphi\rangle$, where $\langle \cdot, \cdot \rangle$ denotes the Hermitian scalar product on S (cf. [1]). One obtains the relations

(8) $$JD = Dj \qquad j\tilde{D} = \tilde{D}j$$

(9) $$j\Omega = \Omega j$$

(10) $$jp_s = p_{m-s} j \qquad (s = 0, \dots, m) ,$$

which imply

(11) $$j\mathscr{D}^{r,s} = \mathscr{D}^{r,m-s} j \quad (r = 1, \dots, m;\ s = 0, \dots, m) .$$

For $\psi \in \Gamma(S)$ let $|\nabla\psi|^2$ and $|\mathscr{D}^r\psi|^2$ be the functions on M^n locally defined by

$$|\nabla\psi|^2 = g^{ik}\langle \nabla_{X_i}\psi, \nabla_{X_k}\psi\rangle, \ |D^r\psi|^2 = g^{ik}\langle D^r_{X_i}\psi, \mathscr{D}^r_{X_k}\psi\rangle .$$

By a straightforward calculation we obtain

PROPOSITION 1. *If* $\psi \in \Gamma(S)$ *is any spinor field, then, for* $r = 1, \dots, m$ *the equation*

$$|\mathscr{D}^r \psi|^2 = |\nabla\psi|^2 + \frac{m-4r}{8r^2}\left(|D\psi|^2 + |\tilde{D}\psi|^2\right) + \frac{1}{8r^2}\left(\langle D\psi, \Omega\tilde{D}\psi\rangle + \langle \Omega\tilde{D}\psi, D\psi\rangle\right) \tag{12}$$

is satisfied. ∎

Now let us assume that M^n is closed. For any spinors $\varphi, \psi \in \Gamma(S)$ let (φ, ψ) denote the Hermitian scalar product defined by

$$(\varphi, \psi) = \int_{M^n} \langle \varphi, \psi \rangle .$$

Moreover, we use the denotation $||\psi||^2 = (\psi, \psi)$ and recall that, in the closed case, the operators D and $\tilde{D}$ are self-adjoint with respect to $(\cdot,\cdot)$. Their spectra lie symmetrically on the real axis and each eigenvalue is of finite multiplicity.

PROPOSITION 2. *If* M^n *is closed, then, for any spinor field* $\psi \in \Gamma(S)$ *and* $r = 1, \dots, m$ *we have the equation*

$$||\mathscr{D}^r \psi||^2 = \frac{2r-1}{2r}\left(D^2\psi - \frac{rR}{4r-2}\psi + \frac{m-2(r-1)}{2r(2r-1)}D^2\psi + \frac{1}{2r(2r-1)}D\tilde{D}\Omega\psi, \psi\right) . \tag{13}$$

Proof. From (1) and (7) we derive the identities

$$D\Omega\tilde{D} = -\tilde{D}\Omega D = D\tilde{D}\Omega + 2D^2 = \Omega D\tilde{D} + 2D^2 . \tag{14}$$

Moreover, by equation (2) for each $\psi \in \Gamma(S)$ we have

$$\langle D^2\psi - \frac{R}{4}\psi, \psi\rangle = \langle \nabla^*\nabla\psi, \psi\rangle .$$

Integrating this equation we obtain

$$\left(D^2\psi - \frac{R}{4}\psi, \psi\right) = ||\nabla\psi||^2 . \tag{15}$$

Using the equations (1), (14), (15) and $D^* = D$, $\tilde{D}^* = \tilde{D}$, $\Omega^* = -\Omega$, equation (13) follows from equation (12) by integration. Q.E.D.

2. THE LOWER BOUND FOR THE FIRST EIGENVALUE OF THE DIRAC OPERATOR

For a real vector X let us consider the corresponding complex vectors $p(X) := \frac{1}{2}(X - iJ(X))$ and $\bar{p}(X) := \frac{1}{2}(X + iJ(X))$. The behaviour of the Clifford multiplication by X, $p(X)$ and $\bar{p}(X)$, respectively, with respect to decomposition (3) is characterized by the formulas (cf. [6])

$$p(X)p_s = p_{s+1}Xp_s \quad \bar{p}(X)p_s = p_{s-1}Xp_s \quad (s \in \mathbb{N}) , \tag{16}$$

where here and in the following the convention $p_s := 0$ for $s \notin \{0, \dots, m\}$ is used. Moreover, let D_+ and D_- be the operators defined by $D_{\pm} = \frac{1}{2}(D_+ \mp i\tilde{D})$. Then by the equations (1) we have

$$D_+^2 = 0 \qquad D_-^2 = 0 . \tag{17}$$

Since, by definition, $D = D_+ + D_-$ and $\tilde{D} = i(D_+ - D_-)$, this yelds

$$D^2 = D_+D_- + D_-D_+ \quad D\tilde{D} = -i(D_+D_-D_-D_+) \tag{18}$$

moreover, the equations (16) imply

$$D_+p_s = p_{s+1}Dp_s \quad D_-p_s = p_{s-1}Dp_s \quad (s \in \mathbb{N}) . \tag{19}$$

Moreover, from [6], [7] we have the following fact:

If $\lambda \neq 0$ is an eigenvalue of the Dirac operator D on a closed Kähler spin manifold $M^n(n = 2m)$, then the corresponding eigenspace $E^\lambda(D)$ splits into the orthogonal direct sum

$$E^\lambda(D) = \oplus_{r=1}^{m} E_r^\lambda(D) \tag{20}$$

where each eigenspinor $\psi \in E_r^\lambda(D)$ has a decomposition $\psi = \psi_{r=1} + \psi_r$ with $\psi_{r-1} \in \Gamma(S_{r-1})$ and $\psi_r \in \Gamma(S_r)$ such that the equations

$$D\psi_{r-1} = \lambda\psi_r \qquad D\psi_r = \lambda\psi_{r-1} \tag{21}$$

$$||\psi_{r-1}|| = ||\psi_r|| \tag{22}$$

are satisfied. Moreover, there exists the relation

$$E_{m-r+1}^\lambda(D) = jE_r^\lambda(D) . \tag{23}$$

Hence, decomposition (20) can be written as

$$E^\lambda(D) = \oplus_{r=1}^{[(m+1)/2]}(E_r^\lambda(D) + jE_r^\lambda(D)) . \tag{24}$$

This gives the motivation for the following definition: an eigenvalue $\lambda \neq 0$ of the Dirac operator is called of type r $(r \in \{1, \dots, [(m+1)/2]\})$ iff $E_r^\lambda(D) \neq 0$.

PROPOSITION 3. *Let M^n be a closed Kähler spin manifold and $\lambda \neq 0$ an eigenvalue of type r of the Dirac operator D, then each eigenspinor $\lambda \in E_r^{\lambda}(D)$ satisfies the equation*

$$(25) \qquad \|\mathscr{D}^r D_-\psi\|^2 = \frac{2r-1}{2r} \int_{M^n} \left(\lambda^2 - \frac{rR}{4r-2}\right) |D_-\psi|^2 .$$

Proof. Let $\psi = \psi_{r-1} + \psi_r \in E_r^{\lambda}(D)$ be an eigenspinor, then, from (19) and (21), we see that $D_-\psi_{r-1} = 0$ and $D_+\psi_r = 0$. Hence, $D_-\psi = D_-\psi_r = D\psi_r = \lambda\psi_{r-1} \neq 0$ is a section of S_{r-1} and satisfies the equations $D_-(D_-\psi) = 0$, $D^2 D_-\psi = \lambda^2 D_-\psi$. Using this we obtain equation (25) from Proposition 2. Q.E.D. ■

Proposition 3 yields immediately

THEOREM 1. *Let M^n be a closed Kähler spin manifold of positive scalar curvature R, then each eigenvalue λ of type r of the Dirac operator D satisfies the inequality*

$$(26) \qquad \lambda^2 \geq \frac{r}{4r-2} R_0 ,$$

where R_0 is the minimum of R on M^n. ■

Comparing the given lower bounds for the different types of eigenvalues from Theorem 1 we obtain

THEOREM 2. *let M^n be a closed K"ahler spin manifold of positive scalar curvature R. If the complex dimension $m = n/2$ is even (odd), then each eigenvalue λ of D satisfies the inequality*

$$(27) \qquad \lambda^2 \geq \frac{1}{4}\frac{m}{m-1} R_0 \qquad \left(\lambda^2 \geq \frac{1}{4}\frac{m+1}{m} R_0\right) .$$

■

From [5] we known that there is a one-to-one correspondence between the spin structures on a Kähler manifold and the equivalence classes $[L]$ of isomorphic holomorphic line bundles L that have the property $L \otimes L = \Lambda^{m,0}$, where $\Lambda^{m,0}$ is the canonical bundle. For such a line bundle L the corresponding spinor bundle is $S_L = \oplus_{r=0}^m \Lambda^{0,r} \otimes L$.

If S_0 is the holomorphic line bundle in decomposition (3), then $[S_0]$ is the spin structure of M^n (cf. Ì8], page 159). For $r = 0, \ldots, m$ we can consider the Laplacians $\Delta_L^{0,r} : \Gamma(\Lambda^{0,r} \otimes L) \to \Gamma(\Lambda^{0,r} \otimes L)$, where $\Delta_L^{0,r}$ denotes the corresponding restriction of the Laplacian Δ_L of the Dolbeault complex of L. The square D^2 of the Dirac operator can be identified with the Laplacian $\Delta_L^{0,\cdot} = \oplus_{r=0}^m \Delta_L^{0,r}$ (cf. [5] or [6], Prop. 9). Hence, Theorem 1 provides

THEOREM 3. *If M^n is a closed Kähler spin manifold of scalar curvature R with $R_0 := \min(R) > 0$, then, for each $L \in [S_0]$ and $r = 1, \ldots, [(m+1)/2]$ $(n = 2m)$, each eigenvalue μ of the Laplacian $\Delta_L^{0,r}$ satisfies the inequality*

$$\mu \geq \frac{r}{4r-2} R_0 .$$

The spectra of $\Delta_L^{0,0}$ and $\Delta_L^{0,1}$ are equal and also for $r = 0, \ldots, m$ the spectra of $\Delta_L^{0,r}$ and $\Delta_L^{0,m-r}$. ∎

3. KÄHLERIAN TWISTOR-SPINORS

A spinor field $\psi \in \Gamma(S)$ is called a Kählerian twistor-spinor of type r or of type (r, s) iff $\psi \in \ker \mathscr{D}^r$ or $\psi \in \ker \mathscr{D}^{r,s}$, respectively.

Such spinors appear in a natural way if one looks at the limiting case of the inequalities (26). In this section we prove some general properties of Kählerian twistor-spinors. We apply the results in the following section, where we consider the limiting case. First of all, we remark that $\psi \in \ker \mathscr{D}^r$ iff ψ satisfies the Kählerian twistor equation of type r

$$\nabla_X \psi + \frac{1}{4r}(X D\psi + J(X)\tilde{D}\psi) = 0 \tag{28}$$

for each vector field X. Using the operators D_+ and D_- this equation can be written

$$\nabla_X \psi + \frac{1}{2r}(\bar{p}(X) D_+ \psi + p(X) D_- \psi) = 0 . \tag{29}$$

Hence, we see that the twistor equation (28) is equivalent to the two equations

$$\nabla_{p(X)} \psi + \frac{1}{2r} p(X) D_- \psi = 0 \tag{30}$$

$$\nabla_{\bar{p}(X)} \psi + \frac{1}{2r} \bar{p}(X) D_+ \psi = 0 . \tag{31}$$

For $r = 1, \ldots, m$ we have $\ker \nabla \subseteq \ker \mathscr{D}^r$, i.e. parallel spinors are twistor-spinors of each type. According to this we say that $\psi \in \ker \mathscr{D}^r$ is trivial iff $\nabla \psi = 0$, and $\ker \mathscr{D}^r$ is called trivial iff $\ker \mathscr{D}^r = \ker \nabla$. We recall that $\mathrm{Ric} \not\equiv 0$ implies $\ker \nabla = 0$ (Ric denotes the Ricci tensor).

PROPOSITION 4. *Each twistor-spinor $\psi \in \ker \mathscr{D}^r$ satisfies the equation*

$$D^2 \psi = \frac{rR}{4r-2} \psi , \tag{32}$$

where R is the scalar curvature.

Proof. Let $P \in M^n$ be an arbitrary point and $(X_1, \dots, X_n)$ a local frame of vector fields in a neighbourhood of P having the property $(\nabla_{X_i} X_k)_P = 0$ for $i, k = 1, \dots, n$. If $\psi \in \ker \mathscr{D}^r$, then we have, for $k = 1, \dots, n$ according to (29),

$$\nabla_{X_k}\psi + \frac{1}{4r}(X_k D\psi + J(X_k)\tilde{D}\psi) = 0 .$$

Applying the operator ∇_{X_i} to this equation in $P \in M^n$ for $i, k = 1, \dots, n$ we obtain

$$\nabla_{X_i}\nabla_{X_k}\psi + \frac{1}{4r}(X_k\nabla_{X_i}D\psi + J(X_k)\nabla_{X_i}\tilde{D}\psi) = 0 .$$

Using this and (1), (2) it follows

$$\begin{aligned} 0 &= g^{ik}\nabla_{X_i}\nabla_{X_k}\psi + \frac{1}{4r}(X^k\nabla_{X_k}D\psi + j(S^k)\nabla_{X_k}\tilde{D}\psi) = \\ &= \nabla^*\nabla\psi + \frac{1}{4r}(D^2\psi + \tilde{D}^2\psi) = -\frac{2r-1}{2r}D^2\psi + \frac{R}{4}\psi . \end{aligned}$$

Q.E.D.

By definition, we have $\ker \mathscr{D}^{r,s} = p_s \ker \mathscr{D}^r \subset \ker \mathscr{D}^r$ and

(33) $$\ker \mathscr{D}^r = \oplus_{s=0}^m \ker \mathscr{D}^{r,s} .$$

Moreover, the relation (11) yields

(34) $$j \ker \mathscr{D}^{r,s} = \ker \mathscr{D}^{r,m-s} .$$

PROPOSITION 5. *Let ψ be a twistor-spinor of type (r, s) that is not trivial, then $s = r - 1$ or $s = m - r + 1$.*

Proof. Let $\psi \in \ker \mathscr{D}^r$ and let $(X_1, \dots, X_n)$ be any local frame, then with (5), it follows from (28) that

$$\begin{aligned} 0 &= X^k\nabla_{X_k}\psi + \frac{1}{4r}(X^k X_k D\psi + X^k J(X_k)\tilde{D}\psi) = \\ &= D\psi + \frac{1}{4r}(-nD\psi - 2\Omega\tilde{D}\psi) = -\frac{1}{2r}((m-2r)D\psi + \Omega\tilde{D}\psi) \end{aligned}$$

and, almost analogously,

$$0 = J(X^k)\nabla_{X_k}\psi + \frac{1}{4r}(J(X^k)X_k D\psi + J(X^k)J(X_k)\tilde{D}\psi) =$$
$$= \frac{1}{2r}(\Omega D\psi - (m-2r)\tilde{D}\psi) .$$

Hence, each $\psi \in \ker \mathscr{D}^r$ satisfies the equations

$$\Omega \tilde{D}\psi = -(m-2r)D\psi \qquad \Omega D\psi = (m-2r)\tilde{D}\psi . \tag{35}$$

Using the relations (7), these equations can be written as follows

$$\tilde{D}\Omega\psi = -(m-2(r-1))D\psi \qquad D\Omega\psi = (m-2(r-1))\tilde{D}\psi . \tag{36}$$

Now we prove that $\ker \mathscr{D}^{r,s} \subseteq \ker \nabla$ for $s \neq r-1,\ m-r+1$. Let $\psi \in \ker \mathscr{D}^{r,s}$, then we have $\Omega\Psi = i(m-2s)\psi$ according to (4), and from (36) we obtain the equations

$$\begin{aligned} &i(m-2s)\tilde{D}\psi = -(m-2(r-1))D\psi \\ &i(m-2s)D\psi = (m-2(r-1))\tilde{D}\psi , \end{aligned} \tag{37}$$

which imply

$$((m-2s)^2 - (m-2(r-1))^2)\tilde{D}\psi = 0$$
$$((m-2s)^2 - (m-2(r-1))^2)D\psi = 0 .$$

Since $s \neq r-1,\ m-r+1$, this yields $D\psi = 0,\ \tilde{D}\psi = 0$ and hence $\nabla\psi = 0$ by the twistor equation (28). Q.E.D. ■

As a corollary of Proposition 5 and (33), (34) we immediately obtain

PROPOSITION 6. *Let M^n be a Kähler spin-manifold that is not Ricci-flat, then, for* $r = 1,\dots,m = n/2$,

$$\ker \mathscr{D}^r = \ker \mathscr{D}^{r,r-1} + j\ker \mathscr{D}^{r,r-1} .$$

■

Using the twistor equation (29) one obtains that, for each $\psi \in \ker \mathscr{D}^r$, the equation

$$\begin{aligned} C(X,Y)\psi = \frac{1}{2r}\big(&\bar{p}(X)\nabla_Y D_+\psi - \bar{p}(Y)\nabla_X D_+\psi + \\ &+ p(X)\nabla_Y D_-\psi - p(Y)\nabla_X D_-\psi\big) \end{aligned} \tag{38}$$

is satisfied, where C denotes the curvature tensor of the spinor bundle S .

LEMMA 7.1. *Let* $\psi \in \ker \mathcal{D}^r$ *, then*

$$(39)\qquad p(\mathrm{Ric}\,(X))\psi = \frac{1}{r}\left(p(X)D_+D_-\psi + 2\nabla_{p(X)}D_-\psi - 2r\nabla_{p(X)}D\psi\right)$$

$$(40)\qquad \bar{p}(\mathrm{Ric}\,(X))\psi = \frac{1}{r}\left(\bar{p}(X)D_-D_+\psi + 2\nabla_{\bar{p}(X)}D_+\psi - 2r\nabla_{\bar{p}(X)}D\psi\right).$$

Proof. Using the equation (38) we obtain, with respect to any local frame $(X_1, \ldots, X_n)$,

$$(*)\qquad \begin{aligned} X^kC(X,X_k)\psi = \frac{1}{2r}\Big(&X^k\bar{p}(X)\nabla_{X_k}D_+\psi - X^k\bar{p}(X_k)\nabla_X D_+\psi + \\ &+ X^kp(X)\nabla_{X_k}D_-\psi - X^kp(X_k)\nabla_X D_-\psi\Big)\,. \end{aligned}$$

From (5) and the complex Clifford relation

$$(41)\qquad \begin{aligned} p(X)\bar{p}(Y) + \bar{p}(Y)p(X) &= -2g(p(X),\bar{p}(Y)) = \\ &- g(X,Y) - i\Omega(X,Y) \end{aligned}$$

we derive the identities

$$(42)\qquad X^k\bar{p}(X_k) = -m - i\Omega \quad X^kp(X_k) = -m + i\Omega\,.$$

Moreover, we have the well-known formula (cf. [2])

$$(43)\qquad X^kC(X,X_k) = -\frac{1}{2}\mathrm{Ric}\,(X)\,.$$

Inserting this into equation (*) we obtain

$$(44)\qquad \begin{aligned} \mathrm{Ric}\,(X)\psi = \frac{1}{r}\big(&\bar{p}(X)D_-D_+\psi + p(X)D_+D_-\psi + \\ &+ 2\nabla_{\bar{p}(X)}D_+\psi + 2\nabla_{p(X)}D_-\psi - 2r\nabla_X D\psi\big)\,. \end{aligned}$$

Since, for a Kähler manifold, $J(\mathrm{Ric}\,(X)) = \mathrm{Ric}\,(J(X))$, equation (44) is equivalent to both the equations (39) and (40). Q.E.D. ■

LEMMA 7.2. *For* $m \neq 2(r-1)$ *, each twistor spinor* $\psi \in \ker \mathcal{D}^{r,r-1}$ *has the property*

$$(45)\qquad D_-\psi = 0$$

and is antiholomorphic.

Proof. If $m \neq 2(r-1)$, the property $D_-\psi = 0$ follows immediately from the equations (37). By equation (30) this implies that ψ is antiholomorphic. Q.E.D. ■

PROPOSITION 7. *For $m \neq 2(r-1)$ each $\psi \in \ker \mathscr{D}^{r,r-1}$ satisfies the equation*

(46) $$\nabla_{p(X)} D\psi = -\frac{1}{2} p(\mathrm{Ric}\,(X))\,\psi$$

(47) $$(r-1)\nabla_{\bar{p}(X)} D\psi = 0$$

(48) $$\bar{p}\left(\mathrm{Ric}\,(X) - \frac{R}{4r-2}X\right)\psi = 0\ .$$

Proof. The equations (39) and (45) imply equation (46). Moreover, the equations (18) and (32) yield

(49) $$D_- D_+\psi = \frac{rR}{4r-2}\psi\ .$$

Inserting this into equation (40) it follows

$$\bar{p}\left(\mathrm{Ric}(X) - \frac{R}{4r-2}X\right)\psi = -2\,\frac{r-1}{r}\nabla_{\bar{p}(X)} D_+\psi\ .$$

By (1è) abd (19) the left-hand side of this equation is an element of $\Gamma(S_{r-2})$ and the right-hand side is an element of $\Gamma(S_r)$. This implies the equations (47) and (48).

Q.E.D. ■

COROLLARY 7.1. *If $m \neq 2(r-1)$ and $r \neq 1$, then, for each $\psi \in \ker \mathscr{D}^{r,r-1}$, the spinor field $D\psi = D_+\psi$ is holomorphic and satisfies the equation*

(50) $$\nabla_X D\psi = -\frac{1}{2} p(\mathrm{Ric}\,(X))\,\psi\ .$$

■

COROLLARY 7.2. *If $m \neq 2(r-1)$ and $\psi \in \ker \mathscr{D}^{r,r-1}$, then*

(51) $$C(X,Y)\psi = -\frac{1}{4r}(\bar{p}(X)p(\mathrm{Ric}\,(Y)) - \bar{p}(Y)p(\mathrm{Ric}\,(X)))\psi\ .$$

COROLLARY 7.3. *If $m \neq 2(r-1)$ and $\psi \in \ker \mathscr{D}^{r,r-1}$, then*

(52) $$\rho\psi = i\frac{R}{4r-2}\psi\ ,$$

where ρ is the Ricci-form.

Proof. If $(X_1, \dots, X_n)$ is any local frame, then one has the identities

$$X^k \mathrm{Ric}\,(X_k) = -R \tag{53}$$

$$J(X^k)\mathrm{Ric}(X_k) = 2\rho\,. \tag{54}$$

Now let $M \neq 2(r-1)$ and $\psi \in \ker \mathscr{D}^{r,r-1}$. Using (42), (53) and (54) equation (48) yields

$$0 = X^k \bar{p}(\mathrm{Ric}\,(X_k))\psi - \frac{R}{4r-2} X^k \bar{p}(X_k)\psi =$$
$$= -i\rho\psi - \frac{R}{4r-2}\psi\,,$$

Q.E.D. ■

PROPOSITION 8. *Each* $\psi \in \ker \mathscr{D}^{r,r-1}$ *satisfies the equations*

$$|\nabla\psi|^2 = \frac{1}{2r}\,|D\psi|^2 \tag{55}$$

$$\Delta\,|\psi|^2 = \frac{R}{4r-2}\,|\psi|^2 - \frac{1}{r}\,|D\psi|^2\;. \tag{56}$$

Proof. For $m \neq 2(r-1)$ and $\psi \in \ker D^{r,r-1}$ we have $D_-\psi = 0$ and, hence $\tilde{D}\psi = iD\psi = iD_+\psi$. It follows that $|\tilde{D}\psi|^2 = |D\psi|^2$ and $\Omega\,\tilde{D}\psi = -(m-2r)D\psi$. Inserting this into equation (12) we obtain (55).

Now let $m = 2(r-1)$. Then the equations (35) yield $\Omega\,\tilde{D}\psi = 2\,D\psi$. Since $\langle D_+\psi, D_-\psi\rangle = 0$, we deduce from $D\psi = D_+\psi + D_-\psi$ and $\tilde{D}\psi = iD_+\psi - iD_-\psi$ the equation $|\tilde{D}\psi|^2 = |D\psi|^2$. Using this we obtain (55) from (12) in this case, too. To prove (56) we remark that (2) and (32) imply

$$\nabla^*\nabla\psi = \frac{R}{8r-4}\psi\,. \tag{57}$$

Moreover, we have the general formula

$$\Delta\,\langle\varphi,\psi\rangle = \langle\nabla^*\nabla\varphi,\psi\rangle + \langle\varphi,\nabla^*\nabla\psi\rangle - 2\,\langle\nabla\varphi,\nabla\psi\rangle\,.$$

Together with (55) and (57) this provides (56). Q.E.D. ■

PROPOSITION 9. *If* M^n *is closed, then a spinor field* $\psi \in \Gamma(S_{r-1})$ *is a twistor-spinor of type* $(r, r-1)$ *iff* ψ *satisfies the two equations*

$$D^2\psi = \frac{rR}{4r-2}\psi \qquad (m - 2(r-1))D_-\psi = 0\,.$$

Proof. For $\psi \in \Gamma(S_{rnl})$ we have with (4), (18) the identities $(m-2(r-1))D^2\psi + D\tilde{D}\Omega\psi = (m-2(r-1))(D^2 + iD\tilde{D})\psi = 2(m-2(r-1))D_+D_-\psi$. Thus from (13), (32) and (45) we see that $\psi \in \Gamma(S_{r-1})$ is a twistor-spinor of type $(r, r-1)$ iff ψ satisfies the equation

$$D^2\psi + \frac{m-2(r-1)}{r(2(r-1)}D_+D_-\psi = \frac{rR}{4r-2}\psi ,$$

Q.E.D. ■

4. THE LIMITING CASE OF THE INEQUALITIES

From Proposition 3 we deduce immediately

THEOREM 4. *let M^n be a closed Kähler spin-manifold of positive scalar curvature R for which $\lambda_0 = \sqrt{rR_0/2(2r-1)}(R_0 = \min(R))$ is an eigenvalue of type r of the Dirac operator D, then M^n is a space of constant scalar curvature $R = R_0$, and for each $\psi \in E_r^{\lambda_0}(D)$ the spinor $D_-\psi$ is a twistor-spinor of type $(r, r-1)$.* ■

Under the suppositions of Theorem 4, by Propositions 7, the Corollaries 7.1, 7.2, 7.3 and Proposition 8 we obtain the following:

If $\psi \in E_r^{\lambda_0}(D)$ and $\psi = \psi_{r-1} + \psi_r$ is the decomposition according to (21), then, for $r \neq 1$,

(58) $$\nabla_X\psi_{r-1} = -\frac{\lambda_0}{2r}\bar{p}(X)\psi_r$$

(59) $$\nabla_X\psi_r = -\frac{1}{2\lambda_0}p(\mathrm{Ric}(X))\psi_{r-1}$$

(60) $$\bar{p}(\mathrm{Ric}(X))\psi_{r-1} = \frac{\lambda_0^2}{r}\bar{p}(X)\psi_{r-1}$$

(61) $$\rho\psi_{r-1} = i\frac{\lambda_0^2}{r}\psi_{r-1} = i\frac{R}{4r-2}\psi_{r-1}$$

(62) $$C(X,Y)\psi_{r-1} = -\frac{1}{4r}(\bar{p}(X)p(\mathrm{Ric}(Y)) - \bar{p}(Y)p(\mathrm{Ric}(X)))\psi_{r-1}$$

(63) $$\Delta\,|\psi_{r-1}|^2 = \frac{\lambda_0^2}{r}\left(|\psi_{r-1}|^2 - |\psi_r|^2\right) .$$

In the special case of $r = 1$ the corresponding nontrivially satisfied equations are

(64) $$\nabla_X\psi_0 = -\frac{\lambda_0}{2}\bar{p}(X)\psi_1$$

(65) $$\nabla_{p(X)}\psi_1 = -\frac{1}{2\lambda_0}\mathrm{Ric}(X)\psi_0$$

(66) $$\Delta\,|\psi_0|^2 = \frac{R}{2}\left(|\psi_0|^2 - |\psi_1|^2\right) .$$

From Theorem 4 we known that, in the limiting case of the inequalities, the scalar curvature R is a positive constant. In the following we show that the basic equations (58), (59), (60) imply additional conditions on the Ricci tensor. First of all we observe that the identities (53) and (54) imply the relations

(67) $$X^k(\nabla_X \mathrm{Ric})(X_k) = 0$$

(68) $$J(X^k)(\nabla_X \mathrm{Ric})(X_k) = 2\nabla_X \rho\,.$$

Moreover, the Ricci form ρ has the properties

(69) $$d\rho = 0 \qquad d^*\rho = 0\ ,$$

where the second of these equations comes from the Bianci identity since R is constant. Using this one derives

(70) $$X^k\left(\nabla_{X_k}\rho\right) = 0 \qquad J(X^k)\left(\nabla_{X_k}\rho\right) = 0\ .$$

Together with (70) the general relation

(71) $$X\rho - \rho X = 2\,\mathrm{Ric}(J(X))$$

yields

(72) $$X^k(\nabla_{X_k}\mathrm{Ric})(J(X)) = -\nabla_X\rho\,.$$

PROPOSITION 10. *For $r > 1$ the equation*

(73) $$\rho\psi_r = -i\frac{\lambda_0^2}{r}\psi_r$$

and for $r > 2$ the equations

(74) $$\bar{p}(\mathrm{Ric}(X))\psi_r = \frac{\lambda_0^2}{r}\bar{p}(X)\psi_r$$

(75) $$\rho\bar{p}(X)\psi_r = i\frac{\lambda_0^2}{r}\bar{p}(X)\psi_r$$

(76) $$\bar{p}((\nabla_X \mathrm{Ric})(Y))\psi_{r-1} = 0$$

(77) $$(\nabla_X\rho)\psi_{r-1} = 0$$

(78) $$\rho\bar{p}(X)\psi_{r-1} = 3i\frac{\lambda_0^2}{r}\bar{p}(X)\psi_{r-1}$$

are satisfied in the limiting case.

Proof. By (60) we have $\bar{p}(\mathrm{Ric}(Y))\psi_{r-1} = \frac{\lambda_0^2}{r}\bar{p}(Y)\psi_{r-1}$. Applying ∇_X to this equation it follows by (58)

$$\bar{p}((\nabla_X \mathrm{Ric})(Y))\psi_{r-1} + \bar{p}\left(\mathrm{Ric}(\nabla_X Y)\right)\psi_{r-1} - $$
$$- \frac{\lambda_0^2}{2r}\bar{p}(\mathrm{Ric}(Y))\,\bar{p}(X)\psi_r = \frac{\lambda_0^2}{r}\bar{p}(\nabla_X Y)\psi_{r-1} - \frac{\lambda_0^3}{2r^2}\bar{p}(Y)\bar{p}(X)\psi_r$$

and hence

$$(*) \qquad \bar{p}\left(\nabla_X \mathrm{Ric})(Y)\right)\psi_{r-1} = -\frac{\lambda_0}{2r}\bar{p}(X)\bar{p}\left(\mathrm{Ric}(Y) - \frac{\lambda_0^2}{r}Y\right)\psi_r .$$

Using (42) and (72) we have by (*)

$$-i\left(\nabla_{\bar{p}(Y)}\rho\right)\psi_{r-1} = X^k\bar{p}\left((\nabla_{X_k}\mathrm{Ric})(Y)\right)\psi_{r-1} =$$
$$= -\frac{\lambda_0^2}{2r}X^k\bar{p}(X_k)\bar{p}\left(\mathrm{Ric}(Y) - \frac{\lambda_0^2}{r}Y\right)\psi_r =$$
$$= \frac{r-1}{r}\lambda_0\bar{p}\left(\mathrm{Ric}(Y) - \frac{\lambda_0^2}{r}Y\right)\psi_r .$$

Thus we obtain

$$(2*) \qquad \left(\nabla_{\bar{p}(Y)}\rho\right)\psi_{r-1} = i\frac{r-1}{r}\lambda_0\bar{p}\left(\mathrm{Ric}(Y) - \frac{\lambda_0^2}{r}Y\right)\psi_r .$$

By (42), (53), (54) and (70) this yields (73). Applying ∇_Y to (61) it follows by (58)

$$(3*) \qquad \left(\nabla_Y\rho\right)\psi_{r-1} = \left(\nabla_{\bar{p}(Y)}\rho\right)\psi_{r-1} = \frac{\lambda_0}{2r}\left(\rho - i\frac{\lambda_0^2}{r}\right)\bar{p}(Y)\psi_r .$$

Using (71) and (73) this assumes the form

$$(4*) \qquad (\nabla_Y\rho)\psi_{r-1} = \left(\nabla_{\bar{p}(Y)}\rho\right)\psi_{r-1} = i\frac{\lambda_0}{r}\bar{p}\left(\mathrm{Ric}(Y) - \frac{\lambda_0^2}{r}Y\right)\psi_r .$$

Comparing (2*) and (4*) we obtain (74), (77) and, hence, (75), (76) by (3*) and (*). Multiplying (61) by $\bar{p}(X)$ and using (71), (60) one proves (78). Q.E.D. ■

Let ∇^r be the metric covariant derivative defined by

$$\nabla_X^r\psi = \nabla_X\psi + \frac{1}{4\lambda_0}\left(\mathrm{Ric}(X) + i(-1)^r J(\mathrm{Ric}(X))L^2\right)\psi$$

with $L = \sum_{s=0}^m i^s p_s$, then we have

COROLLARY 10.1. *For* $r > 2$ *each* $\psi \in E_r^{\lambda_0}(D)$ *is parallel with respect to* ∇^r. *Hence, it satisfies the equation*

$$\nabla_X \psi = -\frac{1}{4\lambda_0}\left(\mathrm{Ric}(X) + i(-1)^r J(\mathrm{Ric}(X))\, L^2\right)\psi \tag{79}$$

for each vector field X *and the function* $|\psi|^2$ *is constant.*

Proof. By (74) the equation (58), (59) can be written in the symmetric form

$$\nabla_X \psi_{r-1} = -\frac{1}{2\lambda_0}\bar{p}(\mathrm{Ric}(X))\,\psi_r \nabla_X \psi_r = -\frac{1}{2\lambda_0} p(\mathrm{Ric}(X))\,\psi_{r-1}\,. \tag{80}$$

These equations are equivalent to $\nabla^r\left(\psi_{r-1} + \psi_r\right) = 0$. Q.E.D. ∎

According to Corollary 10.1 we assume in the following that

$$|\psi|^2 = |\psi_{r-1}|^2 + |\psi_r|^2 = 1\,. \tag{81}$$

By (63) we obtain immediately

COROLLARY 10.2. *If* $r > 2$, *then, for each* $\psi = \psi_{r-1} + \psi_r \in E_r^{\lambda_0}(D)$ *with* $|\psi|^2 = 1$, *the function* $f := |\psi_{r-1}|^2 - \frac{1}{2} = \frac{1}{2} - |\psi_r|^2$ *satisfies the eigenvalue equation*

$$\Delta f = \frac{2\lambda_0^2}{r} f = \frac{R}{2r-1} f\,.$$

PROPOSITION 11. *Let* $r > 2$ *and* $\psi = \psi_{r-1} + \psi_r \in E_r^{\lambda_0}(D)$, *then, for all vector fields* X, Y, *the equations*

$$\bar{p}\left((\nabla_X \mathrm{Ric})(Y)\right)\psi_r = -\frac{1}{\lambda_0} g\left((\mathrm{Ric}^2 - \frac{\lambda_0^2}{r}\mathrm{Ric})(X), \bar{p}(Y)\right)\psi_{r-1} \tag{82}$$

$$\left(\nabla_X \rho\right)\psi_r = -\frac{i}{\lambda_0}\left(\mathrm{Ric}^2 - \frac{\lambda_0^2}{r}\mathrm{Ric}\right)(X)\,\psi_{r-1} \tag{83}$$

are satisfied. Moreover, the Ricci tensor has the property

$$|\mathrm{Ric}|^2 = \frac{R^2}{4r-2} \tag{84}$$

in the limiting case with $r > 2$.

Proof. Applying ∇_Y to equation (74) and using (59), (41), (53) one obtains (82) by an analogous calculation as in the proof of Proposition 10. By (67), (68) and (60) equation (82) yields (83). Finally, with (70), (83) and (53) we have

$$0 = i\lambda_0 X^k \left(\nabla_{X_k}\rho\right)\psi_r =$$
$$= \left(X^k \mathrm{Ric}^2(X_k) - \frac{\lambda_0^2}{r} X^k \mathrm{Ric}(X_k)\right)\psi_{r-1} =$$
$$= \left(-\mathrm{tr}(\mathrm{Ric}^2) + \frac{\lambda_0^2}{r} R\right)\psi_{r-1} = \left(-|\mathrm{Ric}|^2 + \frac{R^2}{4r-2}\right)\psi_{r-1}$$

This implies (84) since supp ψ_{r-1} is dense in M^n. Q.E.D.

THEOREM 5. *In the limiting case with $n = 4r-2$ the manifold M^n is Einstein.*

Proof. If $n = 4r-2$ and $r > 2$, then, by (84),

$$\left|\mathrm{Ric} - \frac{R}{n}\right|^2 = \mathrm{tr}\left(\mathrm{Ric} - \frac{R}{n}\right)^2 = |\mathrm{Ric}|^2 - \frac{R^2}{n} = 0 .$$

In the trivial case of $n = 2$ $(r = 1)$ we have $M^2 = S^2$ and for $n = 6$ $(r = 2)$ our assertion follows from [8]. Q.E.D.

Up to now the problem of classification of all closed Kähler spin maniflds of constant posivite scalar curvture R for which $\lambda_0 = \sqrt{rR|4r-2}$ is the first eivenvalue of type r of the Dirac operator has been solved only for the trivial case $(m,r) = (1,1)$ and for the case $(m,r) = (3,2)$ (cf. [4], [8]). Moreover, according to (61) these spaces can be Einstein only for $r = 1$ or for $m = n/2 = 2r-1$. Hence, for even complex dimension m we have another geometric situation. Except for the dimension $m = 2$, for which only $r = 1$ is possible, in all other even complex dimensions the spaces for which $\sqrt{mR/4(m-1)}$ is the first eivenvalue of the Dirac operator cannot be Einstein.

5. EXAMPLES OF THE LIMITING CASE

For the even complex dimension $m = 2$ we have the example $G_{4,2} = S^2 \times S^2$ since here we have $R = 8$ and the corresponding first eigenvalue of D^2 is equal to 4 (cf. [11]). The following result yields further examples in case of even complex dimension.

THEOREM 6. *Let M^n be a closed Kähler-Einstein spin manifold of odd complex dimension $m = n/2$ and positive scalar curvature R for which $\lambda_0 = \sqrt{(m+1)R/4m}$ is the first eigenvalue of the Dirac operator D, then λ_0 is also the first eivenvalue of* *the Dirac operator $\hat{D}$ on the closed Kähler spin manifold $\hat{M}^{n+2} = M^n \times T^2$, where T^2 denotes the 2-dimensional flat torus.*

Proof. Let $\pi_1 : \hat{M}^{n+2} \to M^n$ and $\pi_2 : \hat{M}^{n+2} \to T^2$ be the natural projections, then we have the decomposition

$$(*) \qquad T\hat{M}^{n+2} = \pi_1^*(TM^n) \oplus \pi_2^*(TT^2) \ .$$

In the following we use the notations $E_1 = \pi_1^*(T^{1,0*}M^n)$, $E_2 = \pi_2^*(T^{1,0*}T^2)$, $\hat{T}^{1,0*} = T^{1,0*}\hat{M}^{n+2}$ and $\hat{\Lambda}^{r,0} = \Lambda^r(\hat{T}^{1,0*})$. According to (*) we have $\hat{T}^{1,0*} = E_1 \oplus E_2$. Using the general formula $\Lambda^k(E_1 \oplus E_2) = \oplus_{i+j=k}(\Lambda^i E_1 \otimes \Lambda^j E_2)$ we find

$$(2*) \qquad \hat{\Lambda}^{r,0} = \Lambda^r E_1 \oplus (\Lambda^{r-1}E_1) \otimes E_2 \ (r = 0, \ldots, m+1) \ .$$

If S_0 is the holomorphic line bundle entering the decomposition (3) of the spinor bundle S of M^n, then $S_0 \otimes S_0 = \Lambda^{m,0}$. Moreover, the spin structure on T^2 is represented by a (trivial) holomorphic line bundle θ with $\theta \otimes \theta = T^{1,0*}T^2$. Using this for $r = m+1$ we obtain, from (2*), the isomorphism $\hat{\Lambda}^{m+1,0} = (\Lambda^m E_1) \otimes E_2 = \hat{L} \otimes \hat{L}$ with $\hat{L} = \pi_1^* S_0 \otimes \pi_2^* \theta$. Hence, the holomorphic line bundle $\hat{L}$ represents the natural induced spin structure on $\hat{M}^{n+2}$. The corresponding spinor bundle is $\hat{S} = \hat{S}_0 \oplus \hat{S}_1 \oplus \ldots \oplus \hat{S}_{m+1}$ with $\hat{S}_r = \hat{\Lambda}^{0,r} \otimes \hat{L}$. Since $S_r = \Lambda^{0,r} \otimes S_0$, (2*) yields $\hat{S}_r = \pi_1^* S_r \otimes \pi_2^* \theta \oplus \pi_1^* S_{r-1} \otimes \pi_2^* \bar{\theta}$ and hence $\hat{S} = \pi_1^* S \otimes \pi_2^* \theta \oplus \pi_1^* S \otimes \pi_2^* \bar{\theta}$. The Clifford multiplication on $\hat{S}$ is given by

$$(3*) \qquad \begin{aligned} X(\psi \otimes \zeta) &= (q(X)\psi) \otimes \zeta + \sqrt{2}|\zeta|^{-2}\zeta^2((1-q)(X))L^2\psi \otimes \bar{\zeta} \\ X(\psi \otimes \bar{\zeta}) &= (q(X)\psi) \otimes \bar{\zeta} - \sqrt{2}|\zeta|^{-2}\zeta^2((1-q)(X))L^2\psi \otimes \zeta \ , \end{aligned}$$

where $X \in T\hat{M}^{n+2}, \psi \in \pi_1^* S, \zeta \in \pi_2^* \theta, \zeta^2 := \zeta \otimes \zeta, L = \sum_{s=0}^m i^s p_S$ and $q : T\hat{M}^{n+2} \to T\hat{M}^{n+2}$ is the projection corresponding to decomposition (*) such that $imq = \pi_1^*(TM^n)$. For $\varphi \in \Gamma(S)$ let $\hat{\varphi} \in \Gamma(\hat{S})$ be defined by $\hat{\varphi} = \pi_1^* \varphi \otimes \pi_2^* \gamma$, where $\gamma \in \Gamma(\theta)$ is a fixed section with $|\gamma| = 1$ and $\nabla\gamma = 0$. Now let $\psi = \psi_{r-1} + \psi_r \in E_r^{\lambda_0}(D)$ be an eigenspinor, then $m = 2r - 1$ and ψ_{r-1}, ψ_r satisfy the equations

$$(4*) \qquad \nabla_Y \psi_{r-1} = -\frac{\lambda_0}{m+1}\bar{p}(Y)\psi_r \quad \nabla_Y \psi_r = -\frac{\lambda_0}{m+1}p(Y)\psi_{r-1}$$

for each vector field Y on M^n (cf. [6]). Since $\nabla\gamma = 0$, from (3*), (4*) we obtain

$$\hat{\nabla}_X \hat{\psi}_{r-1} = -\frac{\lambda_0}{m+1}\bar{p}(qX)\hat{\psi}_r \quad \hat{\nabla}_X \hat{\psi}_r = \frac{\lambda_0}{m+1}p(qX)\hat{\psi}_{r-1}$$

for each vector field X on $\hat{M}^{n+2}$. These equations imply $\hat{D}\hat{\psi}_{r-1} = \lambda_0 \hat{\psi}_r$, $\hat{D}\hat{\psi}_r = \lambda_0 \hat{\psi}_{r-1}$ and hence $\hat{\psi} = \hat{\psi}_{r-1} + \hat{\psi}_r \in E_r^{\lambda_0}(\hat{D})$. Since the scalar curvature of $\hat{M}^{n+2}$

has the positive constant value R, too, it follows by Theorem 2 that λ_0 is the first eigenvalue of $\hat{D}$. Q.E.D. ■

As a corollary of Theorem 6 for the complex dimension $m = 2$ we obtain the additional example $S^2 \times T^2$ and for $m = 4$ the examples $P^3(\mathbb{C}) \times T^2$, $F(\mathbb{C}^3) \times T^2$ since the complex projective space $P^3(\mathbb{C})$ and the flag manifold $F(\mathbb{C}^3)$ are examples of $m = 3$ (cf. [4]).

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